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Note on Taylor's Formula and Some Applications

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In this paper we make some remarks on the generalization of Taylor's formula from S. J. Karlin and W. J. Studden ("Tchebycheff Systems: With Applications in Analysis and Statistics," Interscience, New York, 1966). We also give some applications. © 1987 Academic Press, Inc.

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Let w_i ($i=0, 1,..., n$) be functions of class $C^{n-i}[a, b]$, either positive or negative on $[a, b]$, and let D_j, $j=0, 1,..., n$, denote the first-order differential operator (see [1])

$$(D_j f)(t)=\frac{d}{dt}\left(\frac{f(t)}{w_j(t)}\right).$$

The following is a generalization of the well-known Taylor's formula:

THEOREM 1. *Let* $f\colon [a, b] \to R$ *be a real function such that* $D^n f(x)=(D_n D_{n-1} \cdots D_0 f)(x)$ *is contiuous on* $[a, b]$. *Then*

$$f(t)=\sum_{i=0}^{n} a_i W_i(t; c)+R_n(t) \qquad (\forall t \in [a, b]) \tag{1}$$

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where $c \in [a, b]$,

$$a_i = D^{i-1} f(c)/w_i(c) \qquad (i = 0, 1, \ldots, n;\ D^{-1} f \equiv f), \tag{2}$$

$$W_k(t; x) = w_0(t) \int_x^t w_1(s_1) \int_x^{s_1} w_2(s_2) \cdots \int_x^{s_{k-1}} w_k(s_k)\, ds_k \cdots ds_1 \tag{3}$$

for $k = 1, \ldots, n$ *and* $W_0(t; x) = w_0(t)$; *and*

$$R_n = \int_c^t W_n(t; x)\, D^n f(x)\, dx = D^n f(u) \int_c^t W_n(t; x)\, dx, \tag{4}$$

where $u \in [\min(c, t), \max(c, t)]$.

We also have

$$a_i = \lim_{x \to c} \frac{R_{i-1}}{W_i(x; c)},$$

$$a_i = \lim_{x \to c} \frac{D^{k-1} R_{i-1}(x)}{w_k(x) \int_c^x w_{k+1}(s_{k+1}) \cdots \int_c^{s_{i-1}} w_i(s_i)\, ds_i \cdots ds_{k+1}}; \tag{6}$$

(*for* $k = 0, 1, \ldots, i$ (*for* $k = i$ *we get* (2), *and for* $k = 0$, (5))); *and*

$$a_i = \frac{1}{w_i(c)} \lim_{y \to c} \frac{R_{i-1}(y)}{\int_c^y W_{i-1}(y; s)\, ds}; \tag{7}$$

$$a_i = \frac{1}{w_i(c)} \lim_{y \to c} \frac{D^{k-1} R_{i-1}(y)}{\int_c^y w_k(y) \int_s^y w_{k+1}(s_{k+1}) \cdots \int_s^{s_{i-2}} w_{i-1}(s_{i-1})\, ds_{i-1} \cdots ds_{k+1}\, ds} \tag{8}$$

($k = 0, 1, \ldots, i-1$).

Proof. Applying integration by parts to

$$R_n = \int_c^t W_n(t; x)\, D^n f(x)\, dx = \int_c^t W_n(t; x) \frac{d}{dx}\left(\frac{D^{n-1} f(x)}{w_n(x)}\right) dx,$$

we obtain

$$R_n = -W_n(t; c) \frac{D^{n-1} f(c)}{w_n(c)} - \int_c^t \frac{D^{n-1} f(x)}{w_n(x)} \left(\frac{dW_n(t; x)}{dx}\right) dx$$

$$= -W_n(t; c) \frac{D^{n-1} f(c)}{w_n(c)} + \int_c^t W_{n-1}(t; x)\, D^{n-1} f(x)\, dx,$$

since

$$\frac{1}{w_n(x)}\frac{dW_n(t;x)}{dx}=-W_{n-1}(t;x).$$

Continuing this process we will finally obtain (1) with coefficients a_i given by (2).

On the other hand, (5) could be written in the form

$$a_i=\lim_{x\to c}\frac{R_{i-1}(x)/w_0(x)}{\int_c^x w_1(s_1)\cdots\int_c^{s_{i-1}} w_i(s_i)\,ds_i\cdots ds_1};$$

then, using L'Hospital's rule, we get

$$a_i=\lim_{x\to c}\frac{D_0R_{i-1}(x)}{w_1(x)\int_c^x w_2(s_2)\cdots\int_c^{s_{i-1}} w_i(s_i)\,ds_i\cdots ds_2}.$$

Continuing this process we obtain (6) and, finally,

$$a_i=\lim_{x\to c} D^{i-1}R_{i-1}(x)/w_i(x)=D^{i-1}f(c)/w_i(c).$$

Note, also, that (7) could be written in the form

$$a_i=\frac{1}{w_i(c)}\lim_{y\to c}\frac{R_{i-1}(y)/w_o(y)}{\int_c^y\left(\int_s^y w_1(s_1)\int_s^{s_1} w_2(s_2)\cdots\int_s^{s_{i-2}} w_{i-1}(s_{i-1})\,ds_{i-1}\cdots ds_1\,ds\right.};$$

then, using L'Hospital's rule, we get

$$a_i=\frac{1}{w_i(c)}\lim_{y\to c}\frac{D_0R_{i-1}(y)}{\int_c^y w_1(y)\int_s^y w_2(s_2)\cdots\int_s^{s_{i-2}} w_{i-1}(s_{i-1})\,ds_{i-1}\cdots ds_2\,ds}.$$

Continuing this process we obtain (6) and, finally, for $k=i-1$,

$$a_i=\frac{1}{w_i(c)}\lim_{y\to c}\frac{D^{i-2}R_{i-1}(y)}{\int_c^y w_{i-1}(y)\,ds}=\frac{1}{w_i(c)}\lim_{y\to c}\frac{D^{i-2}R_{i-1}(y)/w_{i-1}(y)}{y-c}$$
$$=D^{i-1}f(c)/w_i(c).$$

Remarks. (1) For $w_0(t)\equiv 1$ and $w_k(t)=k$ $(k=1,\ldots,n)$ we obtain Taylor's formula (and result from [2]).

(2) Theorem 1 is a generalization of some results from [1, pp. 387–389, 454–456].

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Let functions w_i $(i=0, 1, \ldots)$ have all its derivatives on $[a, \infty)$ and let there exist

$$W_i(t) = \lim_{c \to \infty} W_i(t; c),$$

where $W_i(t; c)$ is defined by (3). Then we shall say that

$$f(t) = \sum_{i=0}^{\infty} a_i W_i(t) \tag{9}$$

is a generalized asymptotic series of function f, and the following relations are valid:

$$a_i = \lim_{x \to \infty} R_{i-1}(x)/w_i(x), \tag{10}$$

where

$$R_{i-1}(x) = f(x) - \sum_{k=0}^{i-1} a_k W_k(x)$$

and

$$a_i = \lim_{x \to \infty} \frac{D^{k-1} R_{i-1}(x)}{w_k(x) \int_{\infty}^{x} w_{k+1}(s_{k+1}) \cdots \int_{\infty}^{s_{i-1}} w_i(s_i)\, ds_i \cdots ds_{k+1}}, \tag{11}$$

i.e.,

$$a_i = \lim_{x \to \infty} D^{i-1} f(x)/w_i(x). \tag{12}$$

For $w_0(x) \equiv 1$ and $w_i(x) = -i/x^2$ $(i = 1, 2, \ldots)$, we obtain

$$f(x) = \sum_{k=0}^{\infty} \frac{a_k}{x^k}, \tag{13}$$

where

$$a_n = \lim_{x \to \infty} x^n \left(f(x) - \sum_{k=0}^{n-1} \frac{a_k}{x^k} \right) \qquad (n = 0, 1, \ldots) \tag{14}$$

and

$$a_i = \frac{(i-k)!}{i!} (-1)^k \lim_{x \to \infty} x^{i-k+2} D^{(k-1)} R_{i-1}(x) \qquad (k = 1, \ldots, i) \tag{15}$$

i.e., for $k=i$,

$$a_i=\frac{(-1)^i}{i!}\lim_{x\to\infty}(x^2D^{(i-1)}f(x)), \tag{16}$$

where diferential operator $D^{(k)}f$ is defined recursively by

$$D^{(0)}f(x)=f'(x), \qquad D^{(k)}f(x)=\frac{d}{dx}(x^2D^{(k-1)}f(x)).$$

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Now, suppose that w_i $(i=0, 1,..., n)$ are positive functions of class $C^{n-i}[a, b]$. Then $u_k(t)=W_k(t; a)$ $(k=0, 1,..., n)$ is an ETC-system (extended complete Tchebycheff system). A function f defined on (a, b) is said to be convex with respect to $\{u_i\}_0^n$ if

$$\begin{vmatrix} u_0(t_0) & u_0(t_1) & \cdots & u_0(t_{n+1}) \\ \vdots & \vdots & & \vdots \\ u_n(t_0) & u_n(t_1) & \cdots & u_n(t_{n+1}) \\ f(t_0) & f(t_1) & \cdots & f(t_{n+1}) \end{vmatrix} \geqslant 0$$

for all choices of $\{t_i\}_0^{n+1}$ satisfying $a<t_0<t_1<\cdots<t_{n+1}<b$. In symbols we write $f\in C(u_0, u_1,..., u_n)$ (see [1]).

The following result is a simple consequence of Theorem 1.

THEOREM 2. *Let $f\in C(u_0, u_1,..., u_n)$. If $D^nf(x)$ exists and it is an increasing function on (c, b) with*

$$D^{i-1}f(c)=0 \qquad (i=0, 1,..., n; n\geqslant 1; D^{-1}f\equiv f),$$

then

$$0\leqslant f(t)\leqslant D^nf(t)\int_c^t W_n(t; x)\,dx \qquad (\forall t\in(c, b)).$$

A simple consequence of Theorem 2 is

THEOREM 3. *Let $f\colon [a, b]\to R$ be n times differentiable function on (a, b). If $f(c)=f'(c)=\cdots=f^{(n-1)}(c)=0$, $f^{(n)}(x)(x)>0$ $(\forall x\in(c, b)$, $c>a)$, and if $f^{(n)}(x)$ is an increasing function on (c, b), then*

$$0<f(x)<f^{(n)}(x)\frac{(x-c)^n}{n!} \qquad (\forall x\in(c, b)). \tag{18}$$

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In this part of the paper we shall give two applications of Theorem 3. Note that the following inequalities were proved in [3]:

$$\int_0^x e^{t^2}\,dt > \frac{e^{x^2}}{2x} \quad (x>1) \qquad \text{and} \qquad \int_0^x e^{t^2}\,dt < \frac{3e^{x^2}+x^2-3}{4x} \quad (x>0).$$

Martić [4] proved the following improvements of these results:

$$\int_0^x e^{t^2}\,dt > \frac{1}{2x}\left(e^{x^2} + \sum_{k=2}^{n} \frac{x^{2k}}{k!(2k-1)}\right) \qquad (x \geqslant 1,\ n=2, 3,\ldots)$$

and

$$\int_0^x e^{t^2}\,dt < \frac{1}{3x}\,(2e^{x^2}+x^2-2) \qquad (x>0).$$

Now, we shall prove

THEOREM 4. *The following inequalities are valid for every $x>0$ and $n=1, 2,\ldots$:*

$$0 < \int_0^x e^{t^2}\,dt - \frac{1}{2x}\left(e^{x^2} + \sum_{k=0}^{n} \frac{x^{2k}}{k!(2k-1)}\right) < \frac{x}{2}\left(e^{x^2} - \sum_{k=0}^{n-2} \frac{x^{2k}}{k!}\right), \tag{19}$$

and

$$0 < \frac{1}{3x}\left(2e^{x^2} - \sum_{k=0}^{n} \frac{(k-2)\,x^{2k}}{k!(2k-1)}\right) - \int_0^x e^{t^2}\,dt < \frac{2x^5}{9}\left(e^{x^2} - \sum_{k=0}^{n-3} \frac{x^{2k}}{k!}\right). \tag{20}$$

Proof. Let

$$u(x) = 2x\int_0^x e^{t^2}\,dt - e^{x^2} - \sum_{k=0}^{n} \frac{x^{2k}}{k!(2k-1)}.$$

Then $u'(0)=u(0)=0$ and

$$u''(x) = 2\left(e^{x^2} - \sum_{k=0}^{n-2} \frac{x^{2k}}{k!}\right)$$

is a positive increasing function for $x>0$; so from Theorem 3 in the case $n=2$ and $c=0$, we obtain (19).

Now, let

$$v(x)=2e^{x^2}-\sum_{k=0}^{n}\frac{(k-2)\,x^{2k}}{k!(2k-1)}-3x\int_0^x e^{t^2}\,dt,$$

then $v(0)=v'(0)=v''(0)=0$ and

$$v'''(x)=4x^3\left(e^{x^2}-\sum_{k=0}^{n-3}\frac{x^{2k}}{k!}\right)$$

is a positive increasing function for $x>0$, so, from Theorem 3 in the case $n=3$ and $c=0$, we obtain (20).

In the special case we have

$$0<\int_0^x e^{t^2}\,dt-\frac{1}{2x}(e^{x^2}-1+x^2)<\frac{x}{2}e^{x^2}\qquad (x>0),$$

and

$$0<\frac{1}{3x}(2e^{x^2}+x^2-2)-\int_0^x e^{t^2}\,dt<\frac{2}{9}x^5e^{x^2}\quad (x>0).$$

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